

# ON ALMOST EVERYWHERE EXPONENTIAL SUMMABILITY OF RECTANGULAR PARTIAL SUMS OF DOUBLE TRIGONOMETRIC FOURIER SERIES

USHANGI GOGINAVA AND GRIGORI KARAGULYAN

ABSTRACT. In this paper we study the a.e. exponential strong summability problem for the rectangular partial sums of double trigonometric Fourier series of the functions from  $L \log L$ .

## 1. INTRODUCTION

We denote the set of all non-negative integers by  $\mathbb{N}$ . Let  $\mathbb{T} := [-\pi, \pi) = \mathbb{R}/2\pi$  and  $\mathbb{R} := (-\infty, \infty)$ . Denote by  $L^1(\mathbb{T})$  the class of all measurable functions  $f$  on  $\mathbb{R}$  that are  $2\pi$ -periodic and satisfy

$$\|f\|_1 := \int_{\mathbb{T}} |f| < \infty.$$

The Fourier series of a function  $f \in L^1(\mathbb{T})$  with respect to the trigonometric system is

$$(1) \quad \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where

$$c_n := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-inx} dx$$

are the Fourier coefficients of  $f$ . Denote by  $S_n(x, f)$  the partial sums of the Fourier series of  $f$  and let

$$\sigma_n(x, f) = \frac{1}{n+1} \sum_{k=0}^n S_k(x, f)$$

be the  $(C, 1)$  means of (1). Fejér [1] proved that  $\sigma_n(f)$  converges to  $f$  uniformly for any  $2\pi$ -periodic continuous function. Lebesgue in [18] established almost everywhere convergence of  $(C, 1)$  means if  $f \in L^1(\mathbb{T})$ . The strong

---

<sup>0</sup>2010 Mathematics Subject Classification 42C10.

Key words and phrases: Double Fourier series, strong summability, exponential means.

summability problem, i.e. the convergence of the strong means

$$(2) \quad \frac{1}{n} \sum_{k=0}^{n-1} |S_k(x, f) - f(x)|^p, \quad x \in \mathbb{T}, \quad p > 0,$$

was first considered by Hardy and Littlewood in [9]. They showed that for any  $f \in L^r(\mathbb{T})$  ( $1 < r < \infty$ ) the strong means tend to 0 a.e. as  $n \rightarrow \infty$ . The trigonometric Fourier series of  $f \in L^1(\mathbb{T})$  is said to be  $(H, p)$ -summable at  $x \in \mathbb{T}$  if the values (2) converge to 0 as  $n \rightarrow \infty$ . The  $(H, p)$ -summability problem in  $L^1(\mathbb{T})$  has been investigated by Marcinkiewicz [19] for  $p = 2$ , and later by Zygmund [34] for the general case  $1 \leq p < \infty$ .

Let  $\Phi : [0, \infty) \rightarrow [0, \infty)$ ,  $\Phi(0) = 0$ , be a continuous increasing function. We say a series with the partial sums  $s_n$  strong  $\Phi$ -summable to a limit  $s$  if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi(|s_k - s|) = 0.$$

In [20] Oskolkov first considered the a.e strong  $\Phi$ -summability problem of Fourier series with exponentially growing  $\Phi$ . Namely, he proved a.e strong  $\Phi$ -summability of Fourier series if  $\ln \Phi(t) = O(t / \ln \ln t)$  as  $t \rightarrow \infty$ .

In [21] Rodin proved

**Theorem R** (Rodin). *If a continuous function  $\Phi : [0, \infty) \rightarrow [0, \infty)$ ,  $\Phi(0) = 0$ , satisfies the condition*

$$\limsup_{t \rightarrow +\infty} \frac{\ln \Phi(t)}{t} < \infty,$$

*then for any  $f \in L^1(\mathbb{T})$  the relation*

$$(3) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi(|S_k(x, f) - f(x)|) = 0$$

*holds for a. e.  $x \in \mathbb{T}$ .*

Karagulyan [11, 12] proved that the exponential growth in Rodin's theorem is optimal. Moreover, it was proved

**Theorem K** (Karagulyan). *If a continuous increasing function  $\Phi : [0, \infty) \rightarrow [0, \infty)$ ,  $\Phi(0) = 0$ , satisfies the condition*

$$\limsup_{t \rightarrow +\infty} \frac{\ln \Phi(t)}{t} = \infty,$$

*then there exists a function  $f \in L^1(\mathbb{T})$ , for which the relation*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi(|S_k(x, f)|) = \infty$$

*holds everywhere on  $\mathbb{T}$ .*

In this paper we study the exponential summability problem for the rectangular partial sums of double Fourier series. Let  $f \in L^1(\mathbb{T}^2)$  be a function with Fourier series

$$(4) \quad \sum_{m,n=-\infty}^{\infty} c_{nm} e^{i(mx+ny)},$$

where

$$c_{nm} = \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} f(x_1, x_2) e^{-i(mx_1+nx_2)} dx_1 dx_2$$

are the Fourier coefficients of the function  $f$ . The rectangular partial sums of (4) are defined by

$$S_{MN}(f) = S_{MN}(x_1, x_2, f) = \sum_{m=-M}^M \sum_{n=-N}^N c_{nm} e^{i(mx_1+nx_2)}.$$

We denote by  $L \log L(\mathbb{T}^2)$  the class of measurable functions  $f$ , with

$$\iint_{\mathbb{T}^2} |f| \log^+ |f| < \infty,$$

where  $\log^+ u := \mathbb{I}_{(1,\infty)} \log u$ ,  $u > 0$ . For the rectangular partial sums of two-dimensional trigonometric Fourier series Jessen, Marcinkiewicz and Zygmund [10] has proved for any  $f \in L \log L(\mathbb{T}^2)$  that

$$\lim_{n,m \rightarrow \infty} \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} (S_{ij}(x_1, x_2, f) - f(x_1, x_2)) = 0$$

for a. e.  $(x_1, x_2) \in \mathbb{T}^2$ . They also showed that for every non-negative function  $\omega : [0, \infty) \rightarrow [0, \infty)$  satisfying  $\omega(t) \uparrow \infty$ ,  $\omega(t) (\log^+ t)^{-1} \rightarrow 0$  as  $t \rightarrow \infty$ , there exists a function  $f$  such that  $|f| \omega(|f|) \in L^1(\mathbb{T}^2)$  and the  $(C, 1, 1)$  means of double Fourier series of  $f$  diverge a.e..

The two dimensional a.e. strong rectangular  $(H, p)$ -summability, i.e. the relation

$$\lim_{n,m \rightarrow \infty} \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{ij}(x_1, x_2, f) - f(x_1, x_2)|^p = 0 \text{ a.e.}$$

was proved by Gogoladze [8] for  $f \in L \log L(\mathbb{T}^2)$ . These results show that in two dimensional case the optimal class of functions for  $(C, 1, 1)$  summability and strong summability coincide. That is the class of functions  $L \log L(\mathbb{T}^2)$ .

We prove the following

**Theorem.** *If a continuous increasing function  $\Phi : [0, \infty) \rightarrow [0, \infty)$ ,  $\Phi(0) = 0$ , satisfies the condition*

$$(5) \quad \limsup_{t \rightarrow +\infty} \frac{\ln \Phi(t)}{\sqrt{t / \ln \ln t}} < \infty,$$

then for any  $f \in L \log L (\mathbb{T}^2)$  the relation

$$(6) \quad \lim_{n, m \rightarrow \infty} \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Phi(|S_{ij}(x_1, x_2, f) - f(x_1, x_2)|) = 0$$

holds for a. e.  $(x_1, x_2) \in \mathbb{T}^2$ .

As a corollary of this result we get the Gogoladze [8] theorem on a.e.  $H^p$ -summability of double Fourier series. From Jessen, Marcinkiewicz and Zygmund [10] theorem it follows that the class  $L \log L (\mathbb{T}^2)$  in our theorem is necessary in the context of strong summability question. That is, it is not possible to give a larger convergence space than  $L \log L (\mathbb{T}^2)$ . Our method of proof do not allow to get (6) under the weaker condition

$$(7) \quad \limsup_{t \rightarrow +\infty} \frac{\ln \Phi(t)}{\sqrt{t}} < \infty.$$

There is a conjecture that (7) is the optimal bound of  $\Phi$  ensuring a.e. rectangular strong summability (6) for every function  $f \in L \log L (\mathbb{T}^2)$ .

The results on strong summability and approximation by trigonometric Fourier series have been extended for several other orthogonal systems, see Schipp [23, 24, 25], Leindler [14, 15, 16, 17], Totik [26, 27, 28, 29], Goginava, Gogoladze [5, 6], Goginava, Gogoladze, Karagulyan [7], Gat, Goginava, Karagulyan [3, 4], Weisz [30]-[33].

## 2. AUXILIARY LEMMAS

The notation  $a \lesssim b$  will stand for  $a < c \cdot b$ , where  $c > 0$  is an absolute constant. We shall write  $a \sim b$  if the relations  $a \lesssim b$  and  $b \lesssim a$  hold at the same time. Everywhere below  $q > 1$  will be used as the conjugate of  $p > 1$ , that is  $1/p + 1/q = 1$ .  $[a]$  denotes the integer part of  $a \in \mathbb{R}$ .

The maximal function of a function  $f \in L^1(\mathbb{T})$  is defined by

$$Mf(x) := \sup_{I: x \in I \subset \mathbb{T}} \frac{1}{|I|} \int_I |f(y)| dy,$$

where  $I$  is an open interval. The following one dimensional operators introduced by Gabisonia [2] are significant tools in the investigations of strong summability problems:

$$G_p^{(n)} f(x) := \left( \sum_{k=1}^{[n\pi]} \left( \frac{n}{k} \int_{\frac{k-1}{n}}^{\frac{k}{n}} |f(x+t)| + |f(x-t)| dt \right)^q \right)^{1/q},$$

$$G_p f(x) := \sup_{n \in \mathbb{N}} G_p^{(n)} f(x).$$

Oskolkov's following lemma plays key role in the proof of the basic lemma.

**Lemma 1** (Oskolkov, [20]). *For any family of pairwise disjoint intervals  $\Delta_k \subset \mathbb{T}$  with centers  $c_k$  it holds the inequality*

$$(8) \quad \left| \left\{ x \in \mathbb{T} : \sup_{p>1} \frac{\sum_j \left( \frac{|\Delta_j|}{|x-c_j|+|\Delta_j|} \right)^q}{p \ln \ln(p+2)} > \lambda \right\} \right| \lesssim \exp(-c\lambda), \quad \lambda > 0,$$

where  $c > 0$  is an absolute constant.

One can easily check that

$$\sup_{p>1} \frac{\left( \sum_j \left( \frac{|\Delta_j|}{|x-c_j|+|\Delta_j|} \right)^q \right)^{1/q}}{p \ln \ln(p+2)} \lesssim \left\{ 1, \sup_{p>1} \frac{\sum_j \left( \frac{|\Delta_j|}{|x-c_j|+|\Delta_j|} \right)^q}{p \ln \ln(p+2)} \right\}.$$

Combining this with (8), we get

$$(9) \quad \int_{\mathbb{T}} \sup_{p>1} \frac{\left( \sum_j \left( \frac{|\Delta_j|}{|x-c_j|+|\Delta_j|} \right)^q \right)^{1/q}}{p \ln \ln(p+2)} \lesssim 1.$$

**Lemma 2.** *If  $f \in L^1(\mathbb{T})$ , then*

$$(10) \quad \left| \left\{ x \in \mathbb{T} : \sup_{p>1} \frac{G_p f(x)}{p \ln \ln(p+2)} > \lambda \right\} \right| \lesssim \left( \frac{1}{\lambda} \|f\|_1 \right)^{1/2}, \quad \lambda > 0.$$

*Proof.* It is enough to prove the same estimate for the modified operators

$$(11) \quad G'_p f(x) := \sup_{n \in \mathbb{N}} \left( \sum_{k=1}^{[n\pi]} \left( \frac{n}{k} \int_{\frac{k-1}{n}}^{\frac{k}{n}} |f(x+t)| dt \right)^q \right)^{1/q}.$$

Using the Calderon-Zygmund lemma, for the maximal function we get the relation

$$(12) \quad R_\lambda := \left\{ x \in \mathbb{T} : Mf(x) > \sqrt{\lambda} \right\} = \bigcup_{k=0}^{\infty} \Delta_k, \quad \lambda > 0,$$

where  $\Delta_k \subset \mathbb{T}$  are disjoint open intervals such that

$$(13) \quad \sqrt{\lambda} \leq \frac{1}{|\Delta_k|} \int_{\Delta_k} |f(t)| dt \leq 2\sqrt{\lambda},$$

$$(14) \quad |R_\lambda| \leq \frac{1}{\sqrt{\lambda}} \|f\|_1.$$

Denote  $\delta_k^n := [(k-1)/n, k/n]$  and  $\delta_k^n(x) := x + \delta_k^n$ . Separating the terms in the sum (11) with  $k$  satisfying  $\delta_k^n(x) \subset R_\lambda$ , we get

$$\begin{aligned}
 (15) \quad G'_p f(x) &\leq \sup_{n \in \mathbb{N}} \left( \sum_{k: \delta_k^n(x) \subset R_\lambda} \left( \frac{n}{k} \int_{\frac{k-1}{n}}^{\frac{k}{n}} |f(x+t)| dt \right)^q \right)^{1/q} \\
 &\quad + \sup_{n \in \mathbb{N}} \left( \sum_{k: \delta_k^n(x) \not\subset R_\lambda} \left( \frac{n}{k} \int_{\frac{k-1}{n}}^{\frac{k}{n}} |f(x+t)| dt \right)^q \right)^{1/q} \\
 &:= I + II.
 \end{aligned}$$

From the definition of  $R_\lambda$  in the case  $\delta_k^n(x) \not\subset R_\lambda$  it follows that

$$n \int_{\frac{k-1}{n}}^{\frac{k}{n}} |f(x+t)| dt \leq \sqrt{\lambda}.$$

Thus we conclude

$$(16) \quad II \leq \sqrt{\lambda} \left( \sum_{k=1}^{\infty} \frac{1}{k^q} \right)^{1/q} \lesssim \sqrt{\lambda} \left( \frac{1}{q-1} \right)^{1/q} \lesssim p\sqrt{\lambda}.$$

Given  $x \in \mathbb{T}$  set

$$k_i(x) = \begin{cases} \min \{k : \delta_k^n(x) \subset \Delta_i\} & \text{if } \{k : \delta_k^n(x) \subset \Delta_i\} \neq \emptyset, \\ \infty & \text{if } \{k : \delta_k^n(x) \subset \Delta_i\} = \emptyset. \end{cases}$$

Denote  $\tilde{R}_\lambda := \bigcup_{k=1}^{\infty} 3\Delta_k$  and take an arbitrary point  $x \in \mathbb{T} \setminus \tilde{R}_\lambda$ . One can easily check that if  $k_i(x) \neq \infty$ , then

$$\Delta_i \ni \frac{k_i(x)}{n} \sim |x - c_i|,$$

where  $c_i$  is the center of the interval  $\Delta_i$ . Thus for any  $x \notin \tilde{R}_\lambda$  we obtain

$$\begin{aligned}
 (17) \quad I &= \sup_{n \in \mathbb{N}} \left( \sum_{i=1}^{\infty} \sum_{k: \delta_k^n(x) \subset \Delta_i} \left( \frac{n}{k} \int_{\delta_k^n(x)} |f(t)| dt \right)^q \right)^{1/q} \\
 &\leq \sup_{n \in \mathbb{N}} \left( \sum_{i=1}^{\infty} \left( \sum_{k: \delta_k^n(x) \subset \Delta_i} \frac{n}{k} \int_{\delta_k^n(x)} |f(t)| dt \right)^q \right)^{1/q} \\
 &\leq \sup_{n \in \mathbb{N}} \left( \sum_{i=1}^{\infty} \left( \frac{n |\Delta_i|}{k_i(x) |\Delta_i|} \int_{\Delta_i} |f(t)| dt \right)^q \right)^{1/q} \\
 &\lesssim \sqrt{\lambda} \sup_n \left( \sum_{i=1}^{\infty} \left( \frac{n |\Delta_i|}{k_i(x)} \right)^q \right)^{1/q} \\
 &\lesssim \sqrt{\lambda} \left( \sum_{i=1}^{\infty} \left( \frac{|\Delta_i|}{|x - c_i| + |\Delta_i|} \right)^q \right)^{1/q}, \quad x \notin \tilde{R}_\lambda.
 \end{aligned}$$

Using Chebyshev's inequality, from (9), (16) and (17) it follows that

$$\begin{aligned}
 &\left| \left\{ x \in \mathbb{T} \setminus \tilde{R}_\lambda : \sup_{p>1} \frac{G'_p f(x)}{p \ln \ln(p+2)} > \lambda \right\} \right| \\
 &\lesssim \left| \left\{ x \in \mathbb{T} \setminus \tilde{R}_\lambda : \sqrt{\lambda} \left( 1 + \sup_{p>1} \frac{\left( \sum_j \left( \frac{|\Delta_j|}{|x - c_j| + |\Delta_j|} \right)^q \right)^{1/q}}{p \ln \ln(p+2)} \right) \geq c\lambda \right\} \right| \\
 &\lesssim \frac{1}{\sqrt{\lambda}} \int_{\mathbb{T}} \sup_{p>1} \frac{\left( \sum_j \left( \frac{|\Delta_j|}{|x - c_j| + |\Delta_j|} \right)^q \right)^{1/q}}{p \ln \ln(p+2)} dx \\
 &\lesssim \frac{1}{\sqrt{\lambda}},
 \end{aligned}$$

for an appropriate absolute constant  $c > 0$ . Applying homogeneity, one can get

$$(18) \quad \left| \left\{ x \in \mathbb{T} \setminus \tilde{R}_\lambda : \sup_{p>1} \frac{G'_p f(x)}{p \ln \ln(p+2)} > \lambda \right\} \right| \lesssim \left( \frac{\|f\|_1}{\lambda} \right)^{1/2}, \quad \lambda > 0.$$

Consequently, from (14)-(18) we get

$$\begin{aligned} & \left| \left\{ x \in \mathbb{T} : \sup_{p>1} \frac{G'_p f(x)}{p \ln \ln(p+2)} > \lambda \right\} \right| \\ & \leq \left| \left\{ x \in \mathbb{T} \setminus \tilde{R}_\lambda : \sup_{p>1} \frac{G'_p f(x)}{p \ln \ln(p+2)} > \lambda \right\} \right| + |\tilde{R}_\lambda| \\ & \lesssim \left( \frac{\|f\|_1}{\lambda} \right)^{1/2} + \frac{\|f\|_1}{\sqrt{\lambda}}. \end{aligned}$$

Again using homogeneity, we obtain (10).  $\square$

We will need the following estimations.

**Lemma 3** (Gabisonia, [2]). *If  $p > 1$  and  $f \in L^1(\mathbb{T}^2)$ , then*

$$(19) \quad \left( \frac{1}{n} \sum_{j=0}^{n-1} |S_j(x, f)|^p \right)^{1/p} \lesssim G_p^{(n)} f(x).$$

**Lemma 4** (Schipp, [22]). *If  $f \in L^1(\mathbb{T}^2)$ , then*

$$(20) \quad \left( \frac{1}{n} \sum_{j=0}^{n-1} |S_j(x, f)|^p \right)^{1/p} \lesssim p G_2 f(x).$$

Rodin [21] proved the weak  $(1, 1)$ -type estimate for the operators  $G_p f(x)$  with a fixed  $p > 1$ . From this fact, applying a standard argument, one can derive

**Lemma 5** (Rodin, [21]). *Let  $f \in L \log L(\mathbb{T})$ . Then*

$$\|G_2(f)\|_1 \lesssim 1 + \int_{\mathbb{T}} |f| \log |f|.$$

For any function  $f \in L^1(\mathbb{T}^2)$  define

$$\begin{aligned} G_{p,1}(x_1, x_2; f) &= G_p f_{x_2}(x_1), & G_{p,2}(x_1, x_2; f) &= G_p f_{x_1}(x_2), \\ G_{p,1}^{(n)}(x_1, x_2; f) &= G_p^{(n)} f_{x_2}(x_1), & G_{p,2}^{(n)}(x_1, x_2; f) &= G_p^{(n)} f_{x_1}(x_2), \end{aligned}$$

where  $f_{x_2}(\cdot) = f(\cdot, x_2)$  and  $f_{x_1}(\cdot) = f(x_1, \cdot)$  are considered as functions on  $x_1$  and  $x_2$  respectively. Similarly one dimensional partial sums of  $f(x_1, x_2)$  with respect to each variables will be denoted by

$$S_{n,1}(x_1, x_2, f) = S_n(x_1, f_{x_2}), \quad S_{n,2}(x_1, x_2, f) = S_n(x_2, f_{x_1}).$$



**Lemma 6.** *If  $f \in L \log L(\mathbb{T}^2)$ , then*

$$\left| \left\{ \sup_{p>1} \sup_{n,m \in \mathbb{N}} \frac{\left( \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{i,j}(x_1, x_2, f)|^p \right)^{1/p}}{p^2 \ln \ln(p+2)} > \lambda \right\} \right| \\ \lesssim \left( \frac{1}{\lambda} \left( 1 + \iint_{\mathbb{T}^2} |f| \log^+ |f| \right) \right)^{1/2}, \quad \lambda > 0.$$

*Proof.* Using (19), (20) and generalized Minkowski's inequality, we get

$$\begin{aligned} \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{i,j}(x_1, x_2, f)|^p &= \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{i,1}(x_1, x_2, S_{j,2}(f))|^p \\ &\leq \frac{1}{m} \sum_{j=0}^{m-1} \left( G_{p,1}^{(n)}(x_1, x_2, |S_{j,2}(f)|) \right)^p \\ &\leq \left( G_{p,1}^{(n)} \left( x_1, x_2, \left( \frac{1}{m} \sum_{j=0}^{m-1} |S_{j,2}(f)|^p \right)^{1/p} \right) \right)^p \\ &\leq \left( G_{p,1} \left( x_1, x_2, \left( \frac{1}{m} \sum_{j=0}^{m-1} |S_{j,2}(f)|^p \right)^{1/p} \right) \right)^p \\ &\lesssim p^p (G_{p,1}(x_1, x_2, G_{2,2}(f)))^p. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \Omega &= \left\{ (x_1, x_2) \in \mathbb{T}^2 : \sup_{p>1} \sup_{n,m \in \mathbb{N}} \frac{\left( \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{i,j}(x_1, x_2, f)|^p \right)^{1/p}}{p^2 \ln \ln(p+2)} > \lambda \right\} \\ &\subset \left\{ (x_1, x_2) \in \mathbb{T}^2 : \sup_{p>1} \frac{G_{p,1}(x_1, x_2, G_{2,2}(f))}{p \ln \ln(p+2)} > \lambda \right\}, \end{aligned}$$

then, applying Lemma 2 and 5, we conclude

$$\begin{aligned}
|\Omega| &= \int_{\mathbb{T}^2} \mathbb{I}_\Omega(x_1, x_2) dx_1 dx_2 = \int_{\mathbb{T}} dx_2 \int_{\mathbb{T}} \mathbb{I}_\Omega(x_1, x_2) dx_1 \\
&\lesssim \int_{\mathbb{T}} \left( \frac{1}{\lambda} \int_{\mathbb{T}} G_{2,2}(x_1, x_2, f) dx_1 \right)^{1/2} dx_2 \\
&\lesssim \int_{\mathbb{T}} \left[ \frac{1}{\lambda} \left( 1 + \int_{\mathbb{T}} |f(x_1, x_2)| \log^+ |f(x_1, x_2)| dx_1 \right) \right]^{1/2} dx_2 \\
&\lesssim \left[ \frac{1}{\lambda} \left( 1 + \int_{\mathbb{T}^2} |f(x_1, x_2)| \log^+ |f(x_1, x_2)| dx_1 dx_2 \right) \right]^{1/2}.
\end{aligned}$$

Lemma is proved.  $\square$

### 3. PROOF OF THEOREM 1

Let  $L_M := L_M(\mathbb{T}^2)$  be Orlicz space of functions on  $\mathbb{T}^2$  generated by the Young function  $M(t) = t \log^+ t$ . It is known that  $L_M$  is a Banach space with respect to the Luxemburg norm

$$\|f\|_M := \inf \left\{ \lambda : \lambda > 0, \int_X M\left(\frac{|f|}{\lambda}\right) \leq 1 \right\} < \infty.$$

According to a theorem from ([13], Chap. 2, theorem 9.5) we have

$$0,5 \left( 1 + \int_{\mathbb{T}^2} M(|f|) \right) \leq \|f\|_M \leq 1 + \int_{\mathbb{T}^2} M(|f|)$$

provided  $\|f\|_M = 1$ . Hence from Lemma 6 we conclude

$$(21) \quad \left| \left\{ \sup_{p>1} \sup_{n,m \in \mathbb{N}} \frac{\left( \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{i,j}(f)|^p \right)^{1/p}}{p^2 \log \log(p+2)} > \lambda \right\} \right| \lesssim \left( \frac{\|f\|_M}{\lambda} \right)^{1/2}.$$

Indeed, at first we deduce the case of  $\|f\|_M = 1$ , then using a homogeneity argument, we get the inequality in the general case.

*Proof of Theorem.* First we shall prove that for any  $f \in L \log L(\mathbb{T}^2)$  the relation

$$(22) \quad \lim_{n,m \rightarrow \infty} \sup_{p > 1} \frac{\left( \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{i,j}(f) - f|^p \right)^{1/p}}{p^2 \ln \ln(p+2)} = 0 \text{ a.e. .}$$

Observe that (22) trivially holds for the double trigonometric polynomials. Indeed, let  $T$  be a trigonometric polynomial of degree  $(s_1, s_2)$ . Then we have

$$\begin{aligned} S_{i,j}(T) - T &= 0, \quad i \geq s_1, j \geq s_2, \\ S_{i,j}(T) - T &= S_{s_1,j}(T) - T, \quad i \geq s_1, 0 \leq j < s_2, \\ S_{i,j}(T) - T &= S_{i,s_2}(T) - T, \quad 0 \leq i < s_1, j \geq s_2. \end{aligned}$$

Thus for integers  $n > s_1$  and  $m > s_2$  we have

$$\begin{aligned} & \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{i,j}(T) - T|^p \\ &= \frac{1}{n} \sum_{i=0}^{s_1-1} \frac{1}{m} \sum_{j=s_2}^{m-1} |S_{i,s_2}(T) - T|^p + \frac{1}{m} \sum_{j=0}^{s_2-1} \frac{1}{n} \sum_{i=s_1}^{n-1} |S_{s_1,j}(T) - T|^p \\ & \quad + \frac{1}{nm} \sum_{i=0}^{s_1-1} \sum_{j=0}^{s_2-1} |S_{i,j}(T) - T|^p \\ &\leq \frac{1}{n} \sum_{i=0}^{s_1-1} |S_{i,s_2}(T) - T|^p + \frac{1}{m} \sum_{j=0}^{s_2-1} |S_{s_1,j}(T) - T|^p \\ & \quad + \frac{1}{nm} \sum_{j=0}^{s_2-1} \sum_{i=0}^{s_1-1} |S_{i,j}(T) - T|^p \\ &\leq \frac{c_1}{n} + \frac{c_2}{m}, \end{aligned}$$

where  $c_1$  and  $c_2$  are constants depended on  $T$ . Thus (22) holds if  $f = T$ . To prove the general case it is enough to show that the set

$$G_\lambda = \left\{ \limsup_{n,m \rightarrow \infty} \sup_{p > 1} \frac{\left( \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{i,j}(f) - f|^p \right)^{1/p}}{p^2 \ln \ln(p+2)} > \lambda \right\}$$

has measure zero for any  $\lambda > 0$ . Since  $M(t)$  satisfies the  $\Delta_2$ -condition, the function  $f$  can be approximated by a trigonometric polynomial  $T$  (see [13]), that is

$$\|f - T\|_M < \varepsilon, \quad \|f - T\|_{L^1} < \varepsilon$$

Since (22) holds for  $T$ , applying (21), one can obtain

$$\begin{aligned}
|G_\lambda| &= \left| \left\{ \limsup_{n,m \rightarrow \infty} \sup_{p > 1} \frac{\left( \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{i,j}(f - T) - (f - T)|^p \right)^{1/p}}{p^2 \ln \ln(p+2)} > \lambda \right\} \right| \\
&\leq \left| \left\{ \sup_{n,m \in \mathbb{N}} \sup_{p > 1} \frac{\left( \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{i,j}(f - T)|^p \right)^{1/p}}{p^2 \ln \ln(p+2)} > \lambda/2 \right\} \right| \\
&\quad + \left| \left\{ \sup_{p > 1} \frac{|f - T|}{p^2 \ln \ln(p+2)} > \lambda/2 \right\} \right| \\
&\lesssim \left( \frac{\|f - T\|_M}{\lambda} \right)^{1/2} + \frac{\|f\|_{L^1}}{\lambda} \\
&\leq \left( \frac{\varepsilon}{\lambda} \right)^{1/2} + \frac{\varepsilon}{\lambda}.
\end{aligned}$$

Since  $\varepsilon > 0$  can be taken arbitrarily small, we conclude that  $|G_\lambda| = 0$  for any  $\lambda > 0$  and so (22) holds. To prove (6) observe that

$$\begin{aligned}
(23) \quad u(s) &= \exp \left( \sqrt{\frac{s}{\ln \ln(s+2)}} \right) \\
&\leq v(s) = \sum_{k=1}^{\infty} \left( \frac{d}{k} \sqrt{\frac{s}{\ln \ln(k+2)}} \right)^k, \quad s > 1,
\end{aligned}$$

for some absolute constant  $d$ . Indeed, if  $s \geq 1$ , then one can check that

$$1 < \sqrt{\frac{s}{\ln \ln(s+2)}} < k(s) = \left\lfloor \sqrt{\frac{s}{\ln \ln(s+2)}} \right\rfloor + 1 < 2\sqrt{\frac{s}{\ln \ln(s+2)}},$$

and therefore for enough bigger  $d$  we will have

$$\begin{aligned}
v(s) &\geq \left( \frac{d}{k(s)} \sqrt{\frac{s}{\ln \ln(k(s)+2)}} \right)^{k(s)} \\
&> \left( \frac{d}{2} \sqrt{\frac{\ln \ln(s+2)}{\ln \ln(k(s)+2)}} \right)^{k(s)} > e^{k(s)} \\
&\geq u(s)
\end{aligned}$$

and so (23). If the function  $\Phi$  satisfies (5), then one can check that

$$\Phi(s) \leq \exp \left( \sqrt{\frac{A \cdot s}{\ln \ln(A \cdot s + 2)}} \right) = u(As), \quad s > S,$$

for some positive numbers  $A > 1, S > 1$ . Consider the functions

$$\begin{aligned}\varphi_{i,j}(f) &= S_{i,j}(f) - f, \\ \varphi_{i,j}^*(f) &= \begin{cases} \varphi_{i,j}(f) & \text{if } |\varphi_{i,j}(f)| \leq S, \\ 0 & \text{if } |\varphi_{i,j}(f)| > S, \end{cases} \\ \varphi_{i,j}^{**}(f) &= \varphi_{i,j}(f) - \varphi_{i,j}^*(f).\end{aligned}$$

From (23) and the definition of  $\Phi$  it follows that

$$\begin{aligned}\frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Phi(|\varphi_{i,j}(f)|) &= \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Phi(|\varphi_{i,j}^*(f)|) + \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Phi(|\varphi_{i,j}^{**}(f)|) \\ &\leq \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Phi(|\varphi_{i,j}^*(f)|) + v(A|\varphi_{i,j}^{**}(f)|) \\ &= \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Phi(|\varphi_{i,j}^*(f)|) \\ &\quad + \sum_{k=1}^{\infty} \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left( \frac{d}{k} \sqrt{\frac{A \cdot |\varphi_{i,j}^{**}(f)|}{\ln \ln(k+2)}} \right)^k \\ &\leq \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Phi(|\varphi_{i,j}^*(f)|) \\ &\quad + \sum_{k=1}^{\infty} (d\sqrt{A})^k \left( \sup_{p>1/2} \frac{\left( \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |\varphi_{i,j}(f)|^p \right)^{1/p}}{4p^2 \ln \ln(2p+2)} \right)^{\frac{k}{2}}.\end{aligned}$$

The second term of the last expression tends to zero almost everywhere, since according to (22) we have

$$\begin{aligned}&\limsup_{n,m \rightarrow \infty} \sup_{p>1/2} \frac{\left( \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |\varphi_{i,j}(f)|^p \right)^{1/p}}{4p^2 \ln \ln(2p+2)} \\ &\leq \lim_{n,m \rightarrow \infty} \sup_{p>1} \frac{\left( \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{i,j}(f) - f|^p \right)^{1/p}}{p^2 \ln \ln(p+2)} = 0 \text{ a.e.}\end{aligned}$$

Hence, to prove (6) it is enough to show the same for the first term. From (22) and Chebyshev's inequality it follows that

$$\begin{aligned} r_{n,m}(x_1, x_2) &= \frac{\#\{i, j \in \mathbb{N} : 0 \leq i < n, 0 \leq j < m, \varphi_{i,j}(x_1, x_2) > \varepsilon\}}{nm} \\ &\leq \frac{1}{\varepsilon} \cdot \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |\varphi_{i,j}(x_1, x_2, f)| \rightarrow 0 \text{ a.e.}, \end{aligned}$$

where  $\#C$  denotes the cardinality of a finite set  $C$ . Thus for a.e.  $(x_1, x_2) \in \mathbb{T}^2$  we get

$$\begin{aligned} \limsup_{n,m \rightarrow \infty} \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Phi(|\varphi_{i,j}^*(x_1, x_2, f)|) \\ \leq \limsup_{n,m \rightarrow \infty} (r_{n,m}(x_1, x_2)\Phi(S) + (1 - r_{n,m}(x_1, x_2))\Phi(\varepsilon)) \\ = \Phi(\varepsilon) \text{ a.e.} \end{aligned}$$

Since  $\varepsilon > 0$  can be taken arbitrary small we get

$$\lim_{n,m \rightarrow \infty} \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Phi(|\varphi_{i,j}^*(x_1, x_2, f)|) = 0 \text{ a.e.}$$

and so (6). □

## REFERENCES

- [1] Fejér L., Untersuchungen uber Fouriersche Reihen, Math. Annalen, 58 (1904), 501–569.
- [2] Gabisonia O. D., On strong summability points for Fourier series, Mat. Zametki. 5, 14 (1973), 615–626.
- [3] Gát G., Goginava U., Karagulyan G., Almost everywhere strong summability of Marcinkiewicz means of double Walsh-Fourier series. Anal. Math. 40 (2014), no. 4, 243–266.
- [4] Gát G., Goginava U., Karagulyan G., On everywhere divergence of the strong  $\Phi$ -means of Walsh-Fourier series. J. Math. Anal. Appl. 421 (2015), no. 1, 206–214.
- [5] Goginava U., Gogoladze L., Strong approximation by Marcinkiewicz means of two-dimensional Walsh-Fourier series, Constr. Approx. 35 (2012), no. 1, 1–19.
- [6] Goginava U., Gogoladze L., Strong approximation of two-dimensional Walsh-Fourier series. Studia Sci. Math. Hungar. 49 (2012), no. 2, 170–188.
- [7] Goginava U., Gogoladze L., Karagulyan G., BMO-estimation and almost everywhere exponential summability of quadratic partial sums of double Fourier series. Constr. Approx. 40 (2014), no. 1, 105–120.
- [8] Gogoladze L. D., On strong summability almost everywhere. (Russian) Mat. Sb. (N.S.) 135(177) (1988), no. 2, 158–168, 271; translation in Math. USSR-Sb. 63 (1989), no. 1, 153–16.
- [9] Hardy G. H., Littlewood J. E., Sur la series de Fourier d'une fonction a carre sommable, Comptes Rendus (Paris) 156 (1913), 1307–1309.
- [10] Jessen B., Marcinkiewicz J., Zygmund A., Note on the differentiability of multiple integrals. Fund. Math. 25 (1935), 217–234.
- [11] Karagulyan G. A., On the divergence of strong  $\Phi$ -means of Fourier series, Izv. Acad. Sci. of Armenia, 26(1991), no. 2, 159–162.

- [12] Karagulyan G. A., Everywhere divergent  $\Phi$ -means of Fourier series. (Russian) *Mat. Zametki* 80 (2006), no. 1, 50–59; translation in *Math. Notes* 80 (2006), no. 1-2, 47–56.
- [13] Krasnoselski M. A., Rutitski Ya. B., *Convex functions and Orlicz spaces*, Moscow, 1958 (Russian).
- [14] Leindler L., Über die Approximation im starken Sinne, *Acta Math. Acad. Hungar.* 16 (1965), 255–262.
- [15] Leindler L., On the strong approximation of Fourier series, *Acta Sci. Math. (Szeged)* 38 (1976), 317–324.
- [16] Leindler L., Strong approximation and classes of functions, *Mitteilungen Math. Seminar Giessen*, 132 (1978), 29–38.
- [17] Leindler L., *Strong approximation by Fourier series*, Akademiai Kiado, Budapest, 1985.
- [18] Lebesgue H., Recherches sur la sommabilite forte des series de Fourier, *Math. Annalen* 61 (1905), 251–280.
- [19] Marcinkiewicz J., Sur une methode remarquable de sommation des series doublefes de Fourier. *Ann. Scuola Norm. Sup. Pisa*, 8(1939), 149–160.
- [20] Oskolkov K. I., Strong summability of Fourier series. (Russian) *Studies in the theory of functions of several real variables and the approximation of functions. Trudy Mat. Inst. Steklov.* 172 (1985), 280–290.
- [21] Rodin, V. A., The space BMO and strong means of Fourier series. *Anal. Math.* 16 (1990), no. 4, 291–302.
- [22] Schipp F., On the strong summability of Walsh series. Dedicated to Professors Zoltán Daróczy and Imre Kátai. *Publ. Math. Debrecen* 52 (1998), no. 3-4, 611–633.
- [23] Schipp F., Über die starke Summation von Walsh-Fourier Reihen, *Acta Sci. Math. (Szeged)*, 30 (1969), 77–87.
- [24] Schipp F., On strong approximation of Walsh-Fourier series, *MTA III. Oszt. Kozl.* 19(1969), 101–111 (Hungarian).
- [25] Schipp F., Ky N. X., On strong summability of polynomial expansions, *Anal. Math.* 12 (1986), 115–128.
- [26] Totik V., On the strong approximation of Fourier series. *Acta Math. Acad. Sci. Hungar.* 35 (1980), no. 1-2, 151–172.
- [27] Totik V., On the strong approximation of Fourier series, *Acta Math. Sci. Hungar* 35 (1980), 151–172.
- [28] Totik V., On the generalization of Fejér’s summation theorem, *Functions, Series, Operators; Coll. Math. Soc. J. Bolyai (Budapest) Hungary*, 35, North Holland, Amsterdam-Oxford-New-York, 1980, 1195–1199.
- [29] Totik V., Notes on Fourier series: Strong approximation, *J. Approx. Theory*, 43 (1985), 105–111.
- [30] Weisz F., Strong summability of more-dimensional Ciesielski-Fourier series. *East J. Approx.* 10 (2004), no. 3, 333–354.
- [31] Weisz F., Lebesgue points of double Fourier series and strong summability. *J. Math. Anal. Appl.* 432 (2015), no. 1, 441–462.
- [32] Weisz F., Lebesgue points of two-dimensional Fourier transforms and strong summability. *J. Fourier Anal. Appl.* 21 (2015), no. 4, 885–914.
- [33] Weisz F., Strong summability of Fourier transforms at Lebesgue points and Wiener amalgam spaces. *J. Funct. Spaces* 2015, Art. ID 420750, 10 pp.
- [34] Zygmund A., *Trigonometric series*. Cambridge University Press, Cambridge, 1959.

U. GOGINAVA, DEPARTMENT OF MATHEMATICS, FACULTY OF EXACT AND NATURAL SCIENCES, IVANE JAVAKHISHVILI TBILISI STATE UNIVERSITY, CHAVCHAVADZE STR. 1, TBILISI 0179, GEORGIA

*E-mail address:* zazagoginava@gmail.com

G. A. KARAGULYAN, FACULTY OF MATHEMATICS AND MECHANICS, YEREVAN STATE  
UNIVERSITY, ALEX MANOOGIAN, 1, 0025, YEREVAN, ARMENIA  
*E-mail address:* `g.karagulyan@ysu.am`